Numerical General Relativity

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Overview

1. General Relativity: Some relevant concepts
2. BSSN: Solving the Einstein Equations numerically
3. Analysis Methods
   (black holes, global quantities, gravitational waves)
4. The Einstein Toolkit:
   Open-source software for BSSN, GR Hydro, and friends

Some relevant concepts

GENERAL RELATIVITY

Einstein Equations

- $G_{ab} = 8\pi T_{ab}$

- G_{ab}: Einstein tensor, one measure of curvature
- T_{ab}: stress-energy tensor, describes mass/energy/momentum/pressure/stress densities
- G_{ab} and T_{ab} are symmetric: 10 independent components in 4D

- Loose reading: (some part of) the space-time curvature equals ("is generated by") its matter content

- Not discussing T_{ab} today; will treat it as black box

Spacetime Curvature

- Difference between special and general relativity: in GR, spacetime is *curved*
  - 2D example of a curved manifold: earth's surface
- Can't use a straight coordinate system for a curved manifold!
  - E.g. Cartesian coordinate system doesn't "fit" earth's surface
- In GR, one needs to use curvilinear, time-dependent coordinate systems
- In fact, if one knows how to use arbitrary coordinate systems for a theory (e.g. hydrodynamics or electrodynamics), then working with this theory in GR becomes trivial

Riemann, Ricci, Weyl

- Curvature is measured by *Riemann tensor* R_{abcd}; has 20 independent components in 4D describing curvature completely
- (unrelated to “Riemann problem” in hydrodynamics)

- [blackboard sketch explaining how to measure curvature]

- Can split Riemann tensor into two pieces: *Ricci tensor* R_{ab} and *Weyl tensor* C_{abcd}
  - Ricci tensor directly (locally) determined by matter content (similar to div E and div B in electrodynamics); does not evolve
  - Weyl tensor evolves freely, determined by initial and boundary conditions (describing gravitational interactions)

Metric

- *Metric* g_{ab} is fundamental concept in GR
- "Solving" a spacetime usually means calculating its metric
- Everything can be calculated from the metric
  - Conceptually (within theory of GR) similar to gravitational potential in Newtonian gravity, or potentials Φ/A_i in electrodynamics

- [blackboard sketch explaining meaning of metric]

- Metric g_{ab} is symmetric, has 10 components in 4D
- Metric defines lengths and angles

Covariant Derivatives

- Partial derivatives ∂_a are coordinate dependent
  - In other words, given e.g. a vector field Φ^a, the derivative $\partial_b \Phi^a$ does not transform like a tensor under a coordinate transformation
  - Note: This is unrelated to curvature – just a property when using non-trivial coordinate systems
- Covariant derivatives D_a are coordinate independent ("covariant")
  - $D_b \Phi^a = \partial_b \Phi^a + \Gamma^a{}_{bc} \Phi^c$
  - *Christoffel symbol* $\Gamma^a{}_{bc}$ contains first derivatives of the metric

- A physically meaningful theory must be covariant (coordinate independent)

The Einstein Equations as Wave-Type Equations

- Can express the Riemann tensor, Ricci tensor etc. in terms of the metric:
  - $R_{ab} = -\frac{1}{2} g^{cd} \partial_c \partial_d g_{ab}$ + (many other terms involving first and second derivatives of the metric)
- That is:
  - Einstein equations are 10 wave-type equations (coupled, non-linear) for the metric components
- It is (in principle) well known how to solve such equations (compare Maxwell equations when written in terms of potentials)
- However, the Einstein equations have many terms and are thus technically complex

3+1 Decomposition

- 4-vectors and 4-tensors are very elegant…
- …but our everyday understanding, our astrophysical experience, and most numerical methods are based on space and time being separate…
- …we thus split 4D spacetime into 3D space and 1D time, so that the Einstein equations look like a “conventional” theory.

- Procedure:
  1. Choose some time coordinate t for the spacetime
  2. Hypersurfaces t=const define a 3D space each (must be spacelike)
  3. Split 4-vectors into scalar and 3-vector, 4-tensors into scalar, 3-vector, and 3-tensor

Lapse and Shift

- The 4-metric g_{ab} is split into lapse α, shift β^i, and 3-metric γ_{ij} (with simple geometric meanings)
- Introduce extrinsic curvature K_{ij} of t=const hypersurfaces (essentially time derivative of 3-metric)

- [blackboard sketch explaining lapse and shift]
- [blackboard sketch explaining extrinsic curvature]

  - $g_{00} = -\alpha^2 + \gamma_{ij}\, \beta^i\, \beta^j$
  - $g_{0i} = \gamma_{ij}\, \beta^j$
  - $g_{ij} = \gamma_{ij}$

  - $\partial_t\, \gamma_{ij} = -2\, \alpha\, K_{ij} + D_i\, \beta_j + D_j\, \beta_i$ [D_i is covariant derivative]

Gauge Freedom

- Certain parts of the 3-metric γ_{ij} and extrinsic curvature K_{ij} do not influence curvature
  - They can be chosen freely, and only influence the choice of coordinate system (e.g. div γ_{ij} and trace K_{ij})
  - Called *gauge degrees of freedom*
- Similarly, lapse α and shift β^i are not determined by Einstein equations
  - They select how thegauge degree of freedoms evolve in time
  - Can be chosen freely as well
- 4 gauge degrees of freedom in total

- Compare to choice of gauge freedom (div A) for vector potential A in electrodynamics

Constraints

- It turns out that some of the Einstein equations do not involve time derivatives, i.e. they are not evolution equations
- These have to be satisfied at all times
  - Compare to div E, div B in electrodynamics
- Called *Hamiltonian constraint (energy constraint)* (scalar) and *momentum constraint* (3-vector)
  - Can be cast as elliptic equations (i.e. boundary value problems)
- 4 constraints in total

- Constraints need to be satisfied for initial condition
  - If so, they remain satisfied during time evolution
    - Unless there are numerical errors

ADM Formulation

- A common formulation that casts the 4D Einstein equations into a 3+1 time evolution system
  - ADM: named after Arnowitt, Deser, Misner (1962)
  - Note: several versions of ADM, beware e.g. sign conventions for K_{ij} or R_{ij}

- ADM system:
  - 12 evolved variables: γ_{ij}, K_{ij}
  - 4 gauge variables: α, β^i
  - 4 constraints H, M_i

Literature

- Wald, General Relativity

Solving the Einstein Equations numerically

BSSN

Numerical Stability

- In principle, could now:
  - Write down evolution equations for γ_{ij}, K_{ij} (*ADM system*)
  - Choose some simple gauge conditions α, β^i (e.g. $\alpha=1$, $\beta^i=0$: *normal coordinates*)
  - …and begin to evolve!
- However, it turns out that this is unstable for two (unrelated) reasons:
  - ADM system amplifies small errors in constraints (which are numerically always present)
  - Such simple gauge conditions lead to coordinate singularities (coordinate lines cross after some time)

- Numerical relativity community struggled with this for a long time: hard mathematical problem

BSSN System

- Through trial and error, good evolution systems were found (family of *BSSN formulations*)
- Through mathematical analysis, another good class of evolution systems was found (*harmonic formulations*)
- It turns out that stability crucially depends on the gauge conditions, hence lapse and shift cannot really be chosen freely either

- Literature: Well-posedness of the BSSN system proven (and BSSN equations/gauge conditions listed in detail) in Brown et al., Phys. Rev. D **79**, 044023 (2009)

BSSN Variables

- ADM variables: γ_{ij}, K_{ij}, α, β^i (12 evolved variables)

- BSSN uses a different set of variables (~25 evolved variables):
  - $\phi = {}^1/_{12} \log \det \gamma_{ij}$
  - $\gamma^{\sim}_{ij} = e^{-4\phi} \gamma_{ij}$
  - $K = \gamma^{ij} K_{ij}$
  - $A^{\sim}_{ij} = e^{-4\phi} (K_{ij} - ⅓ \gamma_{ij} K)$
  - $\Gamma^{\sim l} = - \partial_j \gamma^{\sim ij}$
  - α, β^i remain
  - A, B^i are essentially time derivatives of α and β^i

- Note: there are several slightly different BSSN variants; all work fine, but they have slightly different behaviour near black holes

BSSN Gauge Conditions

- Time evolution equations of the BSSN variables are readily determined from their definitions and the ADM evolution equations

- BSSN lapse condition: 1+log slicing
  - Similar to a harmonic time coordinate
  - Also called K driver: drives K (trace of K_{ij}) or $\partial_t K$ to zero
- BSSN shift condition: Γ driver
  - Somewhat similar to harmonic spatial coordinates
  - Drives $\tilde{\Gamma}^{I}$ or $\partial_t \tilde{\Gamma}^{I}$ to zero

- These gauge conditions can drive the coordinate system to being Cartesian (Minkowski) when spacetime is flat

BSSN Constraints

- Since the BSSN system has more variables, it also introduces new constraints (in addition to Hamiltonian and momentum constraint):

  - det $\tilde{\gamma}_{ij} = 1$
  - trace $\tilde{A}_{ij} = 0$
  - $\tilde{\Gamma}^{i} = - \partial_j \tilde{\gamma}^{ij}$ [definition of $\tilde{\Gamma}^{i}$]

- As with the Hamiltonian and momentum constraint, they need to be satisfied initially, and will then be preserved during evolution
  - BSSN contains divergence cleaning terms
- Numerically, trace $\tilde{A}_{ij} = 0$ needs to be enforced explicitly; the other constraints are well behaved on their own

Initial Data Setup

- Metric and extrinsic curvature need to satisfy constraints
- Thus need to solve constraints before evolution – highly non-trivial step!
- Typically, initial data are prepared in terms of ADM variables, then converted to BSSN, then evolved
  - ADM variables are "common language" in numerical relativity (used for initial data, analysis, interaction with GR hydro, etc.)

- *York-Lichnerowicz* procedure for constraint solving:
  - Introduce certain new variables that render constraints elliptic equations
  - Constraints can then be solved with standard methods
  - Most generic way to solve constraints – many other ways exist
  - See e.g. Cook, LRR 2000 5

Singularity Treatment

- Black hole spacetimes contain singularities
  - Loosely speaking, things become infinite at a singularity
  - Mathematically, a singularity is a boundary of the domain, and certain field values may diverge near that boundary
  - Physically, a singularity indicates that a theory is invalid/ inconsistent in this regime
    - E.g. general relativity has not been tested for very high curvature, and one expects that quantum gravity will be necessary to describe this consistently
  - Practically, we have to live with singularities in our spacetimes
- *Cosmic censorship conjecture* states that all singularities will be hidden behind an event horizon, i.e. will not affect anything that can be observed from far away

Singularity Treatment

- Method 1: *Excision*
  - Nothing escapes an event horizon, i.e. event horizon is outflow boundary
  - Cut out a part of the simulation domain inside the event horizon
  - Problem: Black holes move, handling irregularly shaped boundaries is complex
- Method 2: *Punctures*
  - Separate solution into divergent and non-divergent parts
    - E.g. conformal factor
  - Handle divergent part analytically
  - Problem: Black holes move, handling a divergent function analytically imposes inconvenient conditions onto simulation

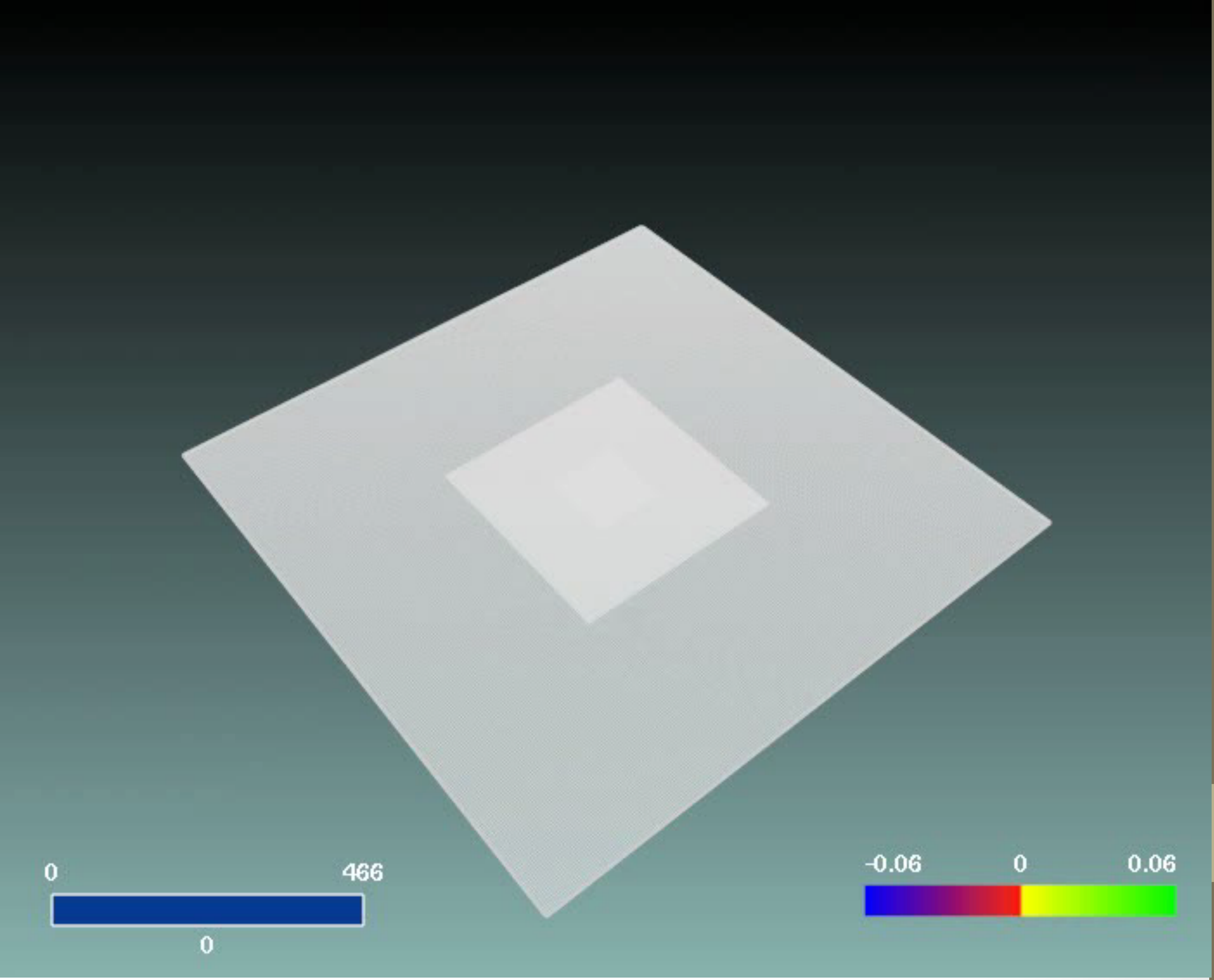
Singularity Treatment

- Method 3: *Numerical dissipation / "moving punctures" / "Turduckening"*
  - Nothing escapes an event horizon, i.e. can violate Einstein equations inside
  - Can e.g. smooth out solution inside (for initial data), or can prevent singularity from forming via dissipation (during time evolution)
  - Works beautifully in practice, proven to be physically correct (since inside EH), was key contribution to successful binary black hole simulations
    - Doesn't seem to work with harmonic formulation

BSSN Summary

- BSSN is a family of evolution systems for the Einstein equations
  - Includes gauge conditions
- Good gauge conditions known
- Can use all initial data (if e.g. available in ADM variables)
  - Can also easily calculate ADM variables, if needed by others
- BSSN has simple way of handling singularities

- Good gauge conditions and singularity handling are probably the largest advantages of BSSN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Black holes, global quantities, gravitational waves

ANALYSIS METHODS

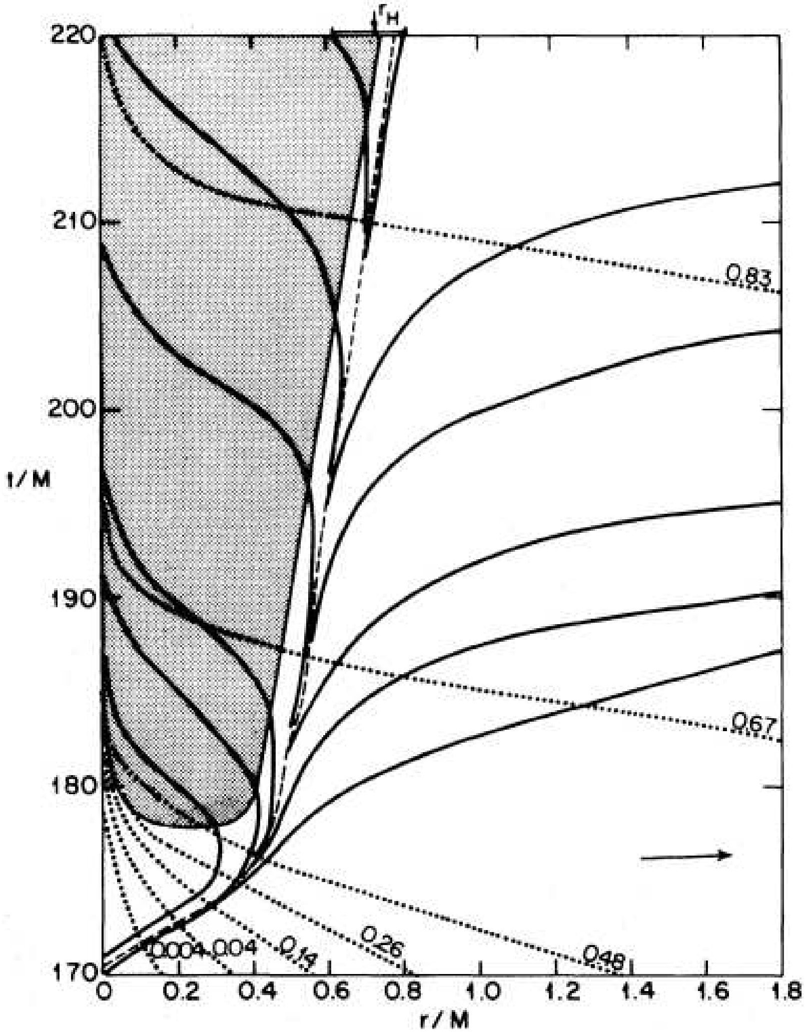
Event Horizons

- It is very difficult to find out where there are black holes in the simulation domain!
  - All one has in the metric (and extrinsic curvature)
- *Event horizons* are a teleological concept and hence often not useful
  - The event horizon is the boundary of the region that can be observed from far away (future null infinity)
  - The location of an event horizon depends only on the future development of the spacetime, not on the past
  - There is no mathematical procedure (nor any kind of measurement) that one could perform to locate an event horizon
    - There could be an event horizon passing through this room right now

Event Horizons

- The location of an event horizon is known iff one knows the future of the spacetime:
  1. In stationary systems
  2. After a numerical time evolution is complete
- Cannot be used to define initial data, cannot be used during time evolution
  - Often, time evolution ends in stationary state

- Procedure for finding event horizons:
  - Event horizon is null surface, made of light rays “moving radially outwards”
  - Start in the future, then trace these light rays backwards in time



Thornburg, LRR 2007 3

Figure 2: *This figure shows part of a simulation of the spherically symmetric collapse of a model stellar core (a $\Gamma = \frac{5}{3}$ polytrope) to a black hole. The event horizon (shown by the dashed line) was computed using the "integrate null geodesics forwards" algorithm described in Section 5.1; solid lines show outgoing null geodesics. The apparent horizon (the boundary of the trapped region, shown shaded) was computed using the zero-finding algorithm discussed in Section 8.1. The dotted lines show the world lines of Lagrangian matter tracers and are labeled by the fraction of baryons interior to them. Figure reprinted with permission from [142].*

Apparent Horizons

- There are many definitions for various kinds of “horizons” in GR
- Most useful probably *apparent horizon* (AH), or *marginally trapped surface* (MTS)
  - Choose a closed 2-surface (distorted sphere)
  - Send light rays outwards
  - Observe how this sphere of light rays grows
  - If its area stays constant (doesn’t grow), it is an MTS
  - Technicality: If there are several MTS, then the outermost is called AH, but terminology is loose
- [blackboard sketch describing MTS]

Apparent Horizons

- Event horizons are a property of the spacetime; they are a global, powerful concept
  - EH is 3D null surface
  - EH begins at point/line, always grows, never ends
- Apparent horizons depend on a choice of foliation (time coordinate), and are thus coordinate dependent
  - AH are 2D surfaces, each defined independently at its own time t, forming a 3D world tube
  - AH world tube can begin/end anytime
  - AH cannot be arbitrarily small
  - We observe that AH world tubes form and annihilate in pairs

- Numerically, finding an AH means solving an elliptic equation, and requires a good initial guess (*horizon tracking*)

Horizon Mass and Angular Momentum

- Can calculate mass and spin of AH, observe how black hole grows
  - (Irreducible) Mass: essentially given by area of AH
    - Will not decrease
  - Angular momentum: more involved, need to find/define approximate Killing vector field (approximate axial symmetry) on horizon
    - Several slightly differing definitions
    - Angular momentum can increase or decrease
    - Will only be conserved if horizon is axially symmetric
  - Total mass: depends on both irreducible mass and angular momentum
    - Can decrease if angular momentum decreases

ADM Quantities

- Can define total mass and angular momentum
  - Cannot just integrate over space: singularities, equivalence principle!
  - Instead, examine metric far away (near infinity)
  - Far away, spacetime will be flat
    - Look closer, will see a perturbation depending on total mass
      - Look closer, will see a perturbation depending on total angular momentum
- Difficult to calculate numerically, since need to evaluate (a) far away from source, and (b) have high accuracy
- Mostly used for initial conditions

Gravitational Waves

- “Gravitational wave” concept not well defined in strong field regime
  - Need to have a *background* (possibly flat) and a *perturbation* to define waves
- Gravitational waves are transverse and have two modes: h^+ and $h^\times$
- Numerically, two possible definitions:
  - Perturbative
  - Curvature based

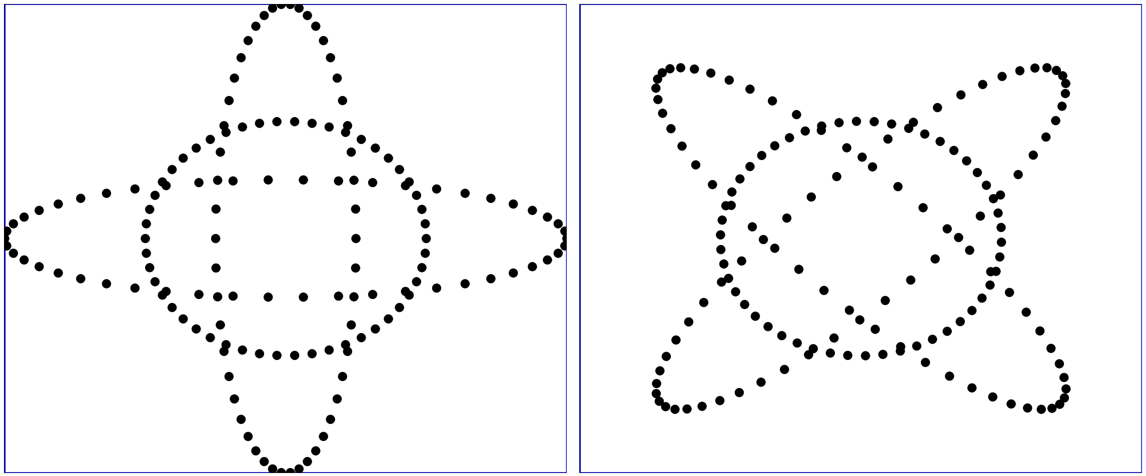


Figure 1: In Einstein's theory, gravitational waves have two independent polarizations. The effect on proper separations of particles in a circular ring in the (x, y)-plane due to a plus-polarized wave traveling in the z-direction is shown in (a) and due to a cross-polarized wave is shown in (b). The ring continuously gets deformed into one of the ellipses and back during the first half of a gravitational wave period and gets deformed into the other ellipse and back during the next half.

Sathyaprakash & Schutz, LRR 2009 2

Gravitational Waves: Perturbative Calculation

- At large distances from a compact object, spacetime will be "simple"
  - will look like a black hole with linear perturbations: gravitational waves
- Procedure: Examine metric far away, decompose into "background" and "perturbation", read off wave content directly
  - Need to know mass, angular momentum of background
  - Need high accuracy

Gravitational Waves: Curvature Based Calculation

- Near infinity, most curvature components will decay away
  - Those decaying most slowly are gravitational waves
  - Fall off with 1/r, i.e. can carry away energy

- Procedure: Examine Weyl tensor far away
  - Decompose it into 5 complex *Weyl scalars* Ψ_n with different fall-off properties
  - Ψ_4 falls off with 1/r
  - Need to integrate to obtain wave content

- Mathematically elegant
  - People say Weyl scalars are coordinate independent, but that is not really true: Weyl scalars depends on choice of tetrad, which people choose depending on coordinates

Open-source software for BSSN, GR Hydro, and friends

THE EINSTEIN TOOLKIT